

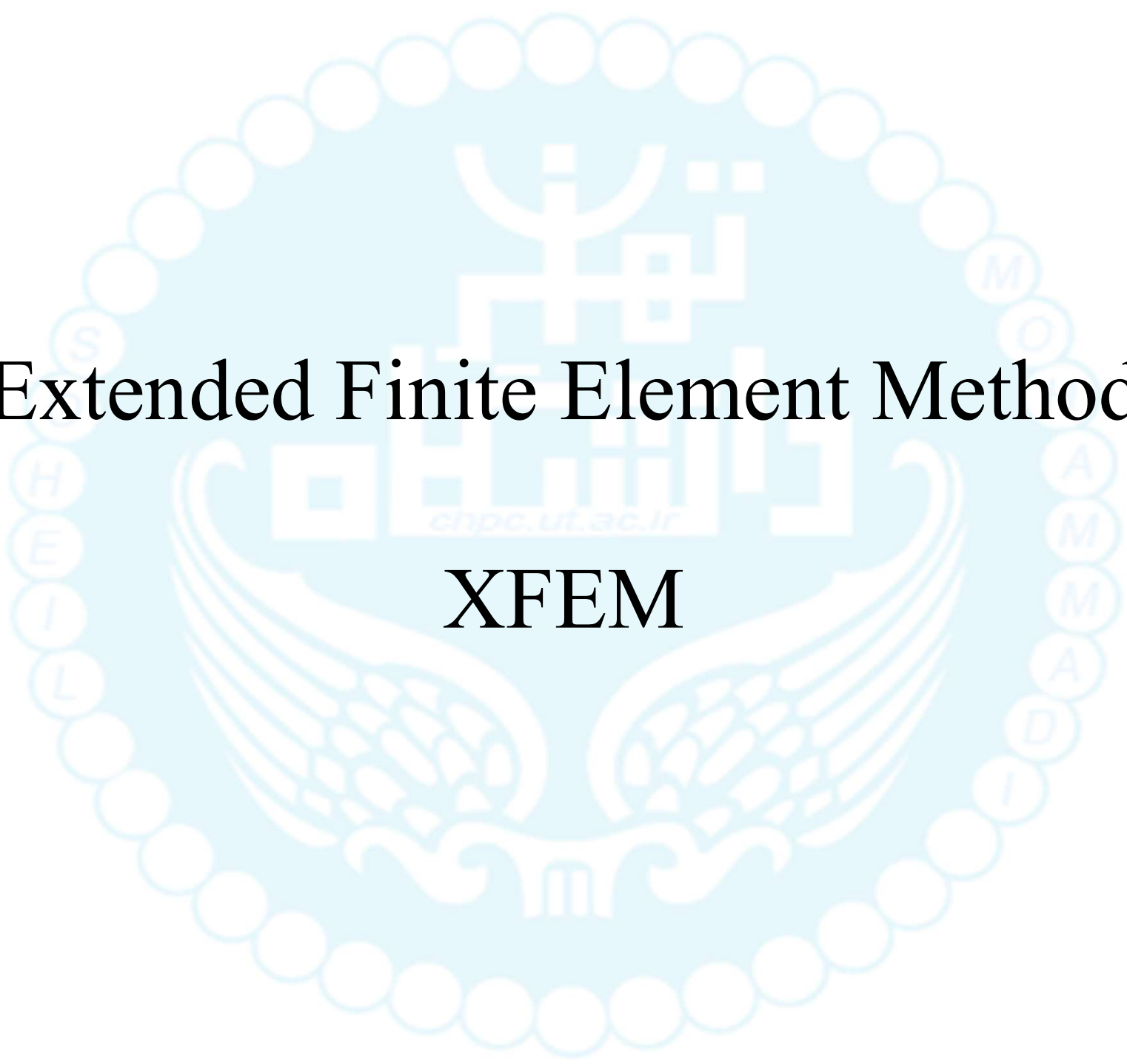


Fracture Mechanics

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Extended Finite Element Method

XFEM



Partition of Unity

ENRICHED APPROXIMATIONS

Enriched approximations have been the subject of several computational studies in the past two decades. Most of them are discussed within the framework of partition of unity.

Partition of unity

Since the introduction of the concept of partition of unity, it has been frequently used in various disciplines as a means of enhancing an original numerical approximation.

A partition of unity (PU) is defined as a set of m functions $f_k(\mathbf{x})$ within a domain Ω_{pu} such that (Melenk and Babuska, 1996),

$$\sum_{k=1}^m f_k(\mathbf{x}) = 1$$

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It can easily be shown that by selection of any arbitrary function $\psi(\mathbf{x})$, partition of unity functions $f_k(\mathbf{x})$ hold the following reproducing property,

$$\sum_{k=1}^m f_k(\mathbf{x})\psi(\mathbf{x}) = \psi(\mathbf{x})$$

This is equivalent to the definition of completeness of a specific order l , where the approximating PU functions $f_k(\mathbf{x})$ can represent exactly the l -order polynomial $\psi(\mathbf{x}) = p^{(l)}(\mathbf{x})$. Then, zero completeness is achieved if Eq. (3.2) holds for a constant $p(\mathbf{x})$.

Isoparametric finite element shape functions, N_j , are the most frequently used PU functions,

$$\sum_{j=1}^n N_j(\mathbf{x}) = 1$$

where n is the number of nodes for each finite element.

The concept of partition of unity, as will be discussed in the next section, provides a mathematical framework for the development of enriched solutions such as the partition of unity finite element method and the extended finite element method.

Intrinsic and extrinsic enrichments

Theoretically, enrichment can be regarded as the principal of increasing the order of completeness or the type of reproduction that can be achieved.

The choice of the enrichment functions depends on the *a priori* solution of the problem. For instance, in a crack analysis this is equivalent to an increase in accuracy of the approximation if analytical near crack tip solutions are somehow included in the enrichment terms.

Computationally, it aims at higher accuracy of the approximation by including the information extracted from the analytical solution into the basic numerical approximation.

Let us begin with the classical approximation of a field variable $\mathbf{u}$ within a finite element method:

$$\mathbf{u} = \sum_{j=1}^n \mathbf{N}_j \bar{\mathbf{u}}_j$$

or in a more appropriate form in terms of the m basis functions $\mathbf{p}$,

$$\mathbf{u} = \mathbf{p}^T \mathbf{a} = \sum_{k=1}^m p_k \mathbf{a}_k$$

where unknowns $\mathbf{a}_k$ are determined from the approximation at nodal points.

The basis function $\mathbf{p}$ may be defined for different orders of completeness:

$$\left\{ \begin{array}{ll} 1D: & \begin{cases} \mathbf{p}^T = \{1, x\} & \text{1st order} \\ \mathbf{p}^T = \{1, x, x^2\} & \text{2nd order} \\ \mathbf{p}^T = \{1, x, x^2, x^3\} & \text{3rd order} \end{cases} \\ 2D: & \begin{cases} \mathbf{p}^T = \{1, x, y\} & \text{1st order} \\ \mathbf{p}^T = \{1, x, y, x^2, xy, y^2\} & \text{2nd order} \\ \mathbf{p}^T = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\} & \text{3rd order} \end{cases} \end{array} \right.$$

The basic idea of the enrichment is to transform the approximation into a more appropriate form to enhance the accuracy of numerical approximation.

The enhancement may be attributed to the degree of consistency of the approximation, or to the capability of approximation to reproduce a given complex field of interest.

There are basically two classes of enrichments:

- enriching the basis function $\mathbf{p}$ (intrinsic enrichment)
- adding enrichment terms to the whole approximation (extrinsic enrichment).

To further elaborate the concept of enrichment, the classical crack problem is considered, and the enrichment concept is explained for this specific case.

The asymptotic near tip displacement field can be written as:

$$u_x = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \left(K_I \cos \frac{\theta}{2} (\kappa - \cos \theta) + K_{II} \sin \frac{\theta}{2} (\kappa + \cos \theta + 2) \right)$$

$$u_y = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \left(K_I \sin \frac{\theta}{2} (\kappa - \cos \theta) - K_{II} \cos \frac{\theta}{2} (\kappa + \cos \theta - 2) \right)$$

where r and θ are the local crack coordinates at the crack tip,, and K_I and K_{II} are the mode I and II stress intensity factors, respectively.

Intrinsic enrichment

In this approach, new terms are added to the basis function $\mathbf{p}$ of approximation (3.5) to satisfy a certain condition of reproducing a complex field (Fries and Belytschko, 2006).

For instance, for a first-order 2D linear basis function

$$\mathbf{p}^{\text{lin}} = \{1, x, y\}$$

new enrichment terms can be included:

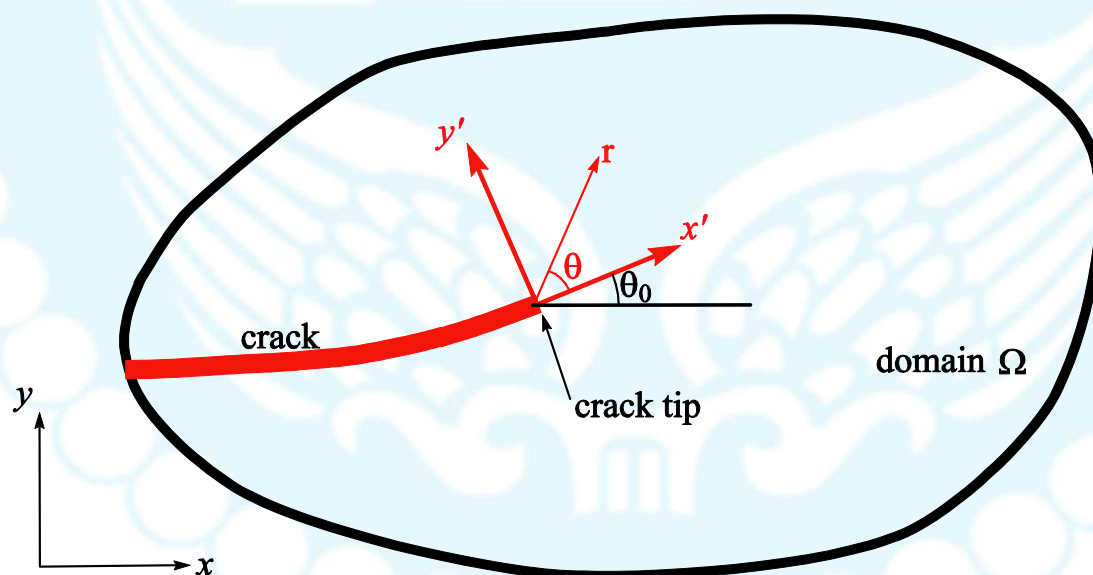
$$\mathbf{p}^{\text{enr}} = \{f_1, f_2, f_3, f_4\}$$

$$\mathbf{p} = \{\mathbf{p}^{\text{lin}}, \mathbf{p}^{\text{enr}}\} = \{1, x, y, f_1, f_2, f_3, f_4\}$$

It can be shown that the asymptotic near crack tip displacement field can be expressed by the following basis function $\mathbf{p}^{\text{enr}}(\mathbf{x})$, defined in the polar coordinate system,

$$\mathbf{p}^{\text{enr}}(\mathbf{x}) = \{f_1, f_2, f_3, f_4\}$$

$$= \left[\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right]$$



The basis function for the total solution must include the constant and linear terms:

$$\mathbf{p}^T(\mathbf{x}) = \left[\underbrace{1, x, y}_{\mathbf{p}^{\text{lin}}(\mathbf{x})}, \underbrace{\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2}}_{\mathbf{p}^{\text{enr}}(\mathbf{x})} \right]$$

which is a familiar basis function previously used for fracture analysis by the meshless element free Galerkin (EFG) method (Belytschko *et al.* 1994),

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{a}(\mathbf{x})$$

where $\mathbf{a}(\mathbf{x})$ is the vector of unknown coefficients, constant or variable, obtained from the weighted least squares (WLS) or moving least squares (MLS) techniques, respectively (Onate *et al.* 1995).

Extrinsic enrichment

Another form of enrichment is based on the so-called extrinsic enrichment, which adds a set of extra terms $\mathbf{p}^{\text{enr}} \mathbf{a}^{\text{enr}}$ to the standard term $\mathbf{p} \mathbf{a}$ to increase the order of completeness.

It can then be defined in the following explicit form:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_k$$

where $\mathbf{a}_k$ are additional unknowns or degrees of freedom associated with the enriched solution.

While in most cases $m=n$, it is not a necessary condition for an extrinsic PU approximation.

In a general partition of unity enrichment, this equation is rewritten as,

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m f_k^{\text{pu}}(\mathbf{x}) \psi(\mathbf{x}) \mathbf{a}_k$$

where $f_k^{\text{pu}}(\mathbf{x})$ are set of the partition of unity functions.

and more generally for more than one enrichment function,

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{k=1}^m f_k^{\text{pu}}(\mathbf{x}) \left[\sum_{l=1}^{m_l} \psi_l(\mathbf{x}) \mathbf{a}_{kl} \right]$$

Partition of unity finite element method

The partition of unity finite element method (PUFEM) (Melenk and Babuska 1996) adopts the same concept of extrinsic enrichment by the use of classical finite element shape functions as the partition of unity functions:

$$f^{\text{pu}}(\mathbf{x}) = N(\mathbf{x})$$

For a general point $\mathbf{x}$ within a finite element,

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \left(\mathbf{u}_j + \sum_{k=1}^m \psi_k(\mathbf{x}) \mathbf{a}_{jk} \right)$$

This approximation is clearly a partition of unity, which guarantees a compatible solution.

Examining the partition of unity finite element approximate solution for a typical enriched node $\mathbf{x}_i$ leads to:

$$\mathbf{u}^h(\mathbf{x}_i) = \sum_{j=1}^n N_j(\mathbf{x}_i) \left(\mathbf{u}_j + \sum_{k=1}^m p_k(\mathbf{x}_i) \mathbf{a}_{jk} \right)$$

where the first part vanishes, except for.

$$N_i(\mathbf{x}_i) \mathbf{u}_i = \mathbf{u}_i$$

Therefore,

$$\mathbf{u}^h(\mathbf{x}_i) = \mathbf{u}_i + \sum_{k=1}^m p_k(\mathbf{x}_i) \mathbf{a}_{ik}$$

which is not a feasible interpolation. $\left(\mathbf{u}^h(\mathbf{x}_i) \neq \mathbf{u}_i \right)$

To satisfy interpolation at nodal points, this equation is transformed to:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \left(\mathbf{u}_j + \sum_{k=1}^m \left(p_k(\mathbf{x}) - p_k(\mathbf{x}_j) \right) \mathbf{a}_j \right)$$

which ensures

$$\mathbf{u}^h(\mathbf{x}_i) = \mathbf{u}_i$$

MLS enrichment

The meshless moving least-square approximation (MLS) can also be enriched in an extrinsic form, as proposed by Duarte and Oden (1995) in the form of the meshless Hp-clouds,

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^{\bar{n}} \Phi_j(\mathbf{x}) \left(\mathbf{u}_j + \sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_{jk}(\mathbf{x}) \right)$$

where $\Phi_j(\mathbf{x})$ are the MLS shape functions evaluated over a moving support domain procedure, $\mathbf{a}_{jk}(\mathbf{x})$ are additional degrees of freedom introduced to enrich the domain of interest, and $\bar{n}$ is the number of nodes within each MLS support domain.

Generalised finite element method

In the generalised finite element method (GFEM), different shape functions are used for the classical and enriched parts of the approximation. Beginning with:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{j=1}^n N_j(\mathbf{x}) \left(\sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_{jk} \right)$$

the generalised form can then be written as:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{j=1}^n \bar{N}_j(\mathbf{x}) \left(\sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_{jk} \right)$$

where $\bar{N}_j(\mathbf{x})$ are the new set of shape functions associated with the enrichment part of the approximation.

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{j=1}^n \bar{N}_j(\mathbf{x}) \left(\sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_{jk} \right)$$

Despite the fact that this approximation does not satisfy the partition of unity condition, it provides some flexibility for the solution by appropriate selection of $\bar{N}$ to enforce specific conditions or constraints.

Extended finite element method

In contrast to PUFEM and GFEM, where the enrichments are usually employed on a global level and over the entire domain, the extended finite element method adopts the same procedure on a local level.

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{j=1}^n N_j(\mathbf{x}) \left(\sum_{k=1}^m p_k(\mathbf{x}) \mathbf{a}_{jk} \right)$$

Assumption of this approximation (3.20) generates a compatible solution even if a local partition of unity is adopted.

This is a considerable computational advantage as it is equivalent to enriching only nodes close to a crack tip; a basis for the extended finite element solution.

Generalised PU enrichment

The original PU enrichment (3.15) can be further generalised if a number of different PU support domains Ω_{pu}^l and associated partition of unity functions $f_k^l(\mathbf{x})$ are used for the enrichment:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j + \sum_{l=1}^{np} \sum_{k=1}^m f_k^l(\mathbf{x}) p^l(\mathbf{x}) \mathbf{a}_k^l$$

where $\mathbf{a}_k^l$ are the additional unknowns associated with each set of np domains of partition of unity.

